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MA111 - Engineering Mathematics - II
Problem Sheet - 3

Ratio and Root tests, Alternating Series, Absolute and Conditional Convergence

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1. Use the Ratio Test to determine if each series converges or diverges.
   (i) $\sum \frac{2^n n!}{n^n}$ (ii) $\sum \frac{n!}{n^n}$ (iii) $\sum \frac{n}{n^2+1} x^n, (x > 0)$
   (iv) $\sum x^n \cos \frac{1}{n}$ (v) $\sum \frac{5^n}{2^n+5}$ (vi) $\sum \sqrt{\frac{n+1}{n^3+1}} x^n$
   (vii) $\sum \frac{n!}{2^{2n-1}}$

2. Use the Root Test to determine if each series converges or diverges.
   (i) $\sum \left(\frac{n}{n+1}\right)^{n^2}$ (ii) $\sum \left(\frac{n+1}{n+2}\right)^n x^n$ (iii) $\sum \frac{n^3}{3^n}$
   (iv) $\sum \left(\frac{n+1}{3n}\right)^n$ (v) $\sum 3^{-2n-5(-1)^n}$ (vi) $\sum_{n=2}^{\infty} \frac{1}{[\log(\log n)]^n}$
   (vii) $\sum \frac{(n-\log n)^n}{2^n n^n}$

3. Let $a_n = \begin{cases} \frac{n}{2^n} & \text{if n is a prime number} \\ \frac{1}{2^n} & \text{otherwise.} \end{cases}$
   Does a_n converge? Give reasons for your answer.

4. Show that Cauchy's root test establishes the convergence of $\sum 3^{-n-(-1)^n}$ while D'Alembert's ratio test fails.

5. Using Leibniz's theorm, determine whether the following series are convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$ (e) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1+2+3+...+n}{n^3}$
   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$ (d) $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\log n}$ (f) $\sum_{n=1}^{\infty} \frac{n+2}{2^n+5}$

6. Which of the following converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

   (a) $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ (e) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$
   (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$ (f) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$

7. Prove that if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} |a_n|$ diverges.

8. Show that the positive terms of the alternating harmonic series form a divergent series (similarly negative terms).

9. Prove or disprove: If $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converges, then $\sum_{n=1}^{\infty} a_n^2 b_n^2$ converges absolutely.

10. Prove that if $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, then the series of its positive terms and the series of its negative term are both convergent.

11. Prove that if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent series, then the series of its positive terms and the series of its negative term are both divergent.

12. Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} a_n$.

13. Show by example that $\sum_{n=1}^{\infty} a_n b_n$ may diverge even if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converges.

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